The Structure of QCD Vacuum and Related Topics *

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Z. Naturforsch. **52a,** 133–136 (1997)

The structure of the vacuum and the attendant state-space for models with a spontaneous breakdown of a global U (1) symmetry is reviewed. QCD is examined in the light of this analysis. Certain commonly held views in QCD are found to need revision and rethinking.

In 1962, Prof. Sudarshan [1] published a paper, in association with his graduate student Thomas F. Jordan, in which the following theorem was proven within the framework of an algebraic formulation of quantum field theory (QFT): the requirement of the uniqueness of the vacuum state implies that the field operators generate an irreducible ring. The same result was proved, using the Wightman formulation of QFT, independently by Borchers [2] and by Ruelle [3]. Additionally, Borchers [2] also proved the converse: irreducibility of fields implies uniqueness of vacuum. Another fundamental result concerns the existence of the linked-cluster expansion for the vacuum expectation values in a QFT with unique vacuum; this was proved by Araki, Hepp, and Ruelle [4]. Earlier, the requirement of a unique vacuum was shown, by Hepp, Jost, Ruelle and Steinman [5], to lead directly to a certain identity for the Wightman functions that is closely related to the linked-cluster expansion. It is interesting to not that this last paper grew out of an attempt to clarify certain issues raised by an article by Prof. Sudarshan and his student K. Bardakci [6]. These results were to play a crucial role in understanding later developments concerning the mathematical structure of theories that display a spontaneous breakdown of an internal symmetry. As will be seen in the sequel, the two aforementioned theorems of QFT also become important in unraveling the structure of quantum chromodynamics (QCD). Before proceeding further we should, however, note one thing. As stated by Jordan and Sudarshan [1], their theorem is true not only in a relativistic QFT but also in a non-relativistic (Galilean-invariant) QFT, since specific properties of Lorentz transformations were not utilized in their derivation. Likewise, Lorentz invariance was not explicitly used (only translational invariance was) in the derivation of the linked-cluster property in [4]. Thus the fundamental result connecting the three properties of uniqueness of vacuum, irreducibility of fields and the linked-cluster property is valid in a relativistic as well as in a non-relativistic QFT. An elementary discussion of this connection was presented by Haag [7].

The first analysis of the structure of a theory with the spontaneous breaking of a symmetry, in the light of the results summarized in the foregoing, was made by Haag [7]. The model was the BCS theory of superconductivity in the limit of infinite volume, where the model is exactly soluble. The underlying symmetry of the model was that of global phase transformations (gauge transformations of the first kind) $\psi \to e^{i\beta}\psi$ acting on the electron field ψ; β is position-independent. The ground state of the system is labeled by an angle $\alpha\,(0 \le \alpha < 2\pi)$, which is the phase of the order parameter. The space of states corresponding to this structure of the ground state is an indexed family $\{H_\alpha\}_{\alpha\in T}$, with T denoting the circle. Any member H_α of this indexed family is a separable Hilbert space with a unique vacuum. Haag showed that in any H_α the field was irreducible, in conformity with the demand of the general result. We might summarize the state of affairs by saying that what we have here is a *gauge-dependent unique vacuum* – a succinct and elegant characterization that we owe to Prof. A. S. Wightman [8]. Moreover, all the Hilbert spaces H_αs have the same physical content. Indeed, an observable is gauge-invariant, it has the structure of a product of equal number of spinor fields (suitably smeared) ψ and the conjugate ψ^* and the expectation value of such an object is independent of the vacuum angle α. Finally,

* Presented at a Workshop in honor of E. C. G. Sudarshan's contributions to Theoretical Physics, held at the University of Texas in Austin, September 15–17, 1991.

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the gauge transformation $\psi \to e^{i\beta}\psi$ corresponds to $H_\alpha \to H_{\alpha+2\beta}$, and this obviously is not a unitary transformation (in a Hilbert space). This lack of unitary implementability is the mathematical statement [9] of spontaneous symmetry breakdown.

Soon after Haag [7], the paper by Araki and Woods [10], on the theory of a Bose gas with finite density in the limit of infinite volume, appeared. Although, one of the main achievements of this paper was to prove the *existence* of an infinite Bose gas, for our purpose what is immediately relevant is to note that the structure of the ground state and the attendant state-space was *exactly* as in the Haag [7] analysis. Here again the gauge transformation (first kind), acting on boson field as $\phi \to e^{i\beta}\phi$, is not implementable as a unitary transformation, e.g., corresponds to $H_\alpha \to H_{\alpha+\beta}$. The vacuum is labeled by the phase α of the order parameter, which is the ground state expectation value of the boson field.

The two non-relativistic models, discussed above, both display a breakdown of a U(1) symmetry. A relativistic QFT model with U(1)-breaking was constructed by Lopuszanski and Reeh [11]; the resulting structure of the vacuum and the attendant state-space was identical with those of the models of Haag [7] and of Araki-Woods [10]. Similar structures appear in many other instances. Examples are the Schwinger model [12], a model due to Ashtekar and Sen [13] where the symmetry breaking arises from the non-trivial topology of (Schwarzschild-Kruskal) space-time, and finally QCD (θ-vacuum).

Let us have a closer look at the mathematical structure of the state-space, which is an indexed family $\{H_\alpha\}_{\alpha\in T}$ of Hilbert spaces. Can the indexed family be interpreted as a single big (non-spearable) Hilbert space? The answer is no. A superposition of two vectors, one taken from the Hilbert space H_α and the other from $H_{\alpha'}$, is *not defined* if $\alpha \neq \alpha'$. Thus the indexed family is not even a linear space. For much the same reason the Hilbert spaces H_αs cannot be interpreted as different sectors associated with a superselection rule. To state the same conclusion somewhat differently, let us recall [14] that a superselection rule is a statement that certain mathematically well-defined states in a Hilbert space do not correspond to physically realizable states. Since a super-position of two vectors, one from H_α and the other from $H_{\alpha'}$, with $\alpha \neq \alpha'$, does not exist to start with, the question of a possible superselection rule does not arise. A valid characterization of the indexed family $\{H_\alpha\}_{\alpha\in T}$ is that it is a Hilbert bundle based on the circle, as shown in [15]. It is a fiber bundle with base space T, the fiber a Hilbert space and the group of the bundle the unitary group (group of unitary operators) of the Hilbert space. Notice that our bundle is not a trivial product space $H \otimes T$. A Hilbert bundle is a product space if the group of the bundle is trivial (one element group). However, in the sense of fiber bundle theory, our bundle is equivalent, *in the group of the bundle*, to a product. This follows from a general result of Borchers and Sen [16] to the effect that any Hilbert bundle with a paracompact base space is equivalent to a product. Finally, in the bundle language the unitarily non-implementable transformations, such as those corresponding to $H_\alpha \to H_{\alpha+\beta}$, can be given precise mathematical meaning as bundle mappings.

The structure of the vacuum and the concomitant state-space for U(1)-broken theories detailed above, is only half the story. To see the other half, as unraveled in [7, 10 and 11] we proceed as follows. We consider the *direct integral*, in the sense of Von Neuman, of Hilbert spaces H_α with respect to the normalized (Lebesgue) measure on the circle T. The result is a certain Hilbert space K. In the bundle language, elements of K are simply the cross-sections of the Hilbert bundle. Now, the Hilbert space K is reducible in the sense that the fields are represented reducibly in K and there exists a set Ω_n of *degenerate* vacua, labeled by the set Z of integers, in K. This may be seen as follows. Denote by Ω_0 the direct integral of the α-vacuum states $|\alpha\rangle$

$$|\Omega_0\rangle = \int^{\oplus} |\alpha\rangle\,. \tag{1}$$

One checks that $|\Omega_0\rangle$ is normalized, $\langle\Omega_0|\Omega_0\rangle = 1$, as was $|\alpha\rangle$, $\langle\alpha|\alpha\rangle = 1$. Consider the operator U in K, which has the direct integral decomposition

$$U = \int^{\oplus} e^{-i\alpha} I_\alpha\,, \tag{2}$$

where I_α is the identity operator in H_α. It follows

$$\langle\Omega_0|U|\Omega_0\rangle = \int_0^{2\pi} e^{-i\alpha}\langle\alpha|\alpha\rangle \frac{d\alpha}{2\pi} = 0\,. \tag{3}$$

The first step in the above is just the definition of the direct integral. We interpret the above result to imply that the state $|\Omega_1\rangle = U|\Omega_0\rangle$ in K, is orthogonal to $|\Omega_0\rangle$. By repeated application, one gets the string $|\Omega_n\rangle = U^n|\Omega_0\rangle$ of mutually orthogonal vacuum states, all translationally invariant, in the Hilbert space K. Let us finally recall the meaning of a direct integral. Let the states $\phi, \psi \in K$ and the operator A in

K have the decompositions

$$|\phi\rangle = \int^{\oplus} |\phi_\alpha\rangle , \quad A = \int^{\oplus} A_\alpha . \tag{4}$$

Then the direct integral means

$$\langle\phi|A|\psi\rangle = \int_0^{2\pi} \langle\phi_\alpha|A_\alpha|\psi_\alpha\rangle \frac{d\alpha}{2\pi} . \tag{5}$$

It is easy to see that the operator U commutes with all elements of the ring generated by the fields. The 'differential component' $U_\alpha = e^{-i\alpha} 1_\alpha$ of U obviously commutes with all operators in H_α, and from it the desired conclusion follows by use of (2) in the appropriate direct integral. But U is a non-trivial operator in K (not a c-number). Thus K must be reducible. The space K breaks up into sectors K_n, where K_n has the vacuum $|\Omega_n\rangle$ and *observables* map each K_n into itself. These sectors can be interpreted as the coherent sectors of a superselection rule whose existence again expresses the underlying gauge-invariance of the theory. The expectation value of an obversable in a state in K_n is independent of n; thus the $|\Omega_n\rangle$, although mathematically distinct, all have the same physical content. The physical origin of the degeneracy of the ground state was explained thus by Araki and Woods [10]: the operator U creates a zero-momentum boson-like excitation; in an infinite system the addition (or subtraction) of a finite number of such excitations cannot alter the underlying physical content.

Structure of the QCD Vacuum

How much of the foregoing structure survives in QCD? It is instructive to approach this problem in three stages. Fist consider a pure gauge (local) theory. There is the string of degenerate vacua Ω_n indexed by the integer that labels the elements of the homotopy group π_3 (SU (3)) of the gauge group SU (3) color. The operator U in K is just the generator of the third homotopy group. The space K possesses a direct integral decomposition leading the the θ-vacuum. Thus it appears that what we have here is a situation exactly analogous to that in U (1)-broken theories, where the U (1) symmetry now is, precisely, the Pontryagin dual to the homotopy group π_3 (SU (3)). However, there is a new feature which seems to spoil the analogy. As a result of tunneling between Ω_n vacua, the θ-vacuum energy acquires a band structure [17, 18]; the θ-vacuum energy-density is proportional to $\cos\theta$.

In the second stage we introduce fermions and neglect the fermion masses. Now we have a gauge-variant conserved axial current. The U (1) symmetry can now be identified with the chiral symmetry. The vacuum structure continues to be the same as in a pure gauge theory; but now the band structure in the θ-vacuum energy has disappeared [17, 18]. We now have a genuine U (1)-broken theory.

In the third and final stage, we lift the restriction of massless fermions and let quarks have finite masses. The vacuum structure continues to be exactly the same as before but finite quark masses lead to two effects. First, the band structure of the θ-vacuum energy reappears; secondly a new phenomenon – that of strong CP-violation – emerges.

Thus, the θ-vacuum has an energy band structure, except when quarks are massless. Let us now scrutinize this piece of conventional wisdom in QCD. First of all, the presence of a vacuum energy violates the time-translational invariance. Next, the vacuum *energy-density* is finite and negative, and thus the θ-vacuum energy must tend to negative infinity if the vacuum is invariant under spatial translations. Of course, one can say that the θ-vacuum is *not* translationally invariant. The conclusion, then, would have to be that QCD is *inconsistent with* QFT (recall that translational invariance is part of the definition of the vacuum state [19], and there would be nothing further to discuss. On the other hand, an infinite vacuum energy is not all that startling a feature in field theory. We have the familiar cases of the infinite zero-point energy of a boson field and the infinite energy of a filled Dirac sea for a fermion field. And the present situation can be handled in much the same way. That is, by simply dropping the infinite contribution by a redefinition of the zero of energy. Now the θ-vacuumis translationally invariant, and tuneling disappears.

A technical remark ought to be made. It is customary in QCD to represent the θ-vacuum state as

$$|\theta\rangle = \sum_n e^{in\theta} |\Omega_n\rangle \tag{6}$$

as an improper (non-normalizable) vector of the Hilbert space K. This is hardly the desired representation for $|\theta\rangle$, which is a normalized vector in the Hilbert space H_θ. The correct formalism, using the theory of direct integration, automatically preserves the desired normalization properties.

Strong CP Violation in QCD

Explicit calculations show [20, 21] that the expectation value of the electric dipole moment operator D_θ

for the neutron, in the θ-sector, is proportional to $\sin\theta$ (all quarks massive). From this, we can compute the corresponding expectation value in the space K (say, in the $|\Omega_0\rangle$ sector of K). The result is

$$\langle D\rangle_{\Omega_0} \sim \int_0^{2\pi} \sin\theta \frac{d\theta}{2\pi} = 0 . \quad (7)$$

Thus strong CP violation is absent in the reducible Hilbert space K! What conclusion can one draw concerning the electric dipole moment of real life neutrons? Nothing much it would seem, unless we are able to decide as to which Hilbert space, out of the possibilities H_θ and K, corresponds to the space of *physical* states. Does QCD provide a criterion to select out of these two alternatives? It has been claimed that H_θ is the space of physical states while K is not, because energy is diagonal in θ-vacuum and not so in Ω_n. We have already examined this claim in the foregoing and found it to be false. It is claimed [17] that K is unphysical since the linked-cluster property is not valid there. But why should this be the criterion to select physical states? Look again at the models of [7] and [10] here it is the space K that is the space of physical states [the expectation value of every gauge-variant operator vanishes in K, thus the states correspond most closely to those labeled by the eigenvalues of (a complete set of commuting) observables]. In any theory of the sort that we have been discussing, there is a very good reason behind the θ sector. In any Hilbert space, H_θ there is a unique vacuum, the fields irreducible and linked-cluster property is valid. H_θ is thus the arena of a *conventional* field theory, where uniqueness of vacuum is one of the axioms [19]; we can apply usual rules of field theory there. The reducible Hilbert space K has none of these properties. But it does not follow from this that H_θ is *necessarily* the space of physical states. Thus one cannot decide, on theoretical grounds, if QCD does or does not predict a strong CP violation. Perhaps the issue could be settled experimentally. Finally, note that the electric charge $e\theta/2\pi$ for dyons [22] average out to $e/2$ in the Ω_n sector!

Some of the issues dealt with in this article have been raised in a recent communication from Sachs [23]. As should be evident to the reader, there are important differences between the treatment of these questions in [23] as compared to the present article.

Acknowledgement

Prof. E. C. G. Sudarshan has been my friend, philosopher and guide for the past quarter century and more. It gives me pleasure to dedicate this article to him on the occasion of his *Festschrift*.

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